AD-A194 535 CONVEX DUALITY APPROACH TO THE OPTIMAL CONTROL OR DIFFUSIONS..(U) BROWN UNIV PROVIDENCE RI LEFSCHETZ CENTER FOR DYNAMICAL SYSTE.. W H FLEMING ET AL. JAN 88 1/1

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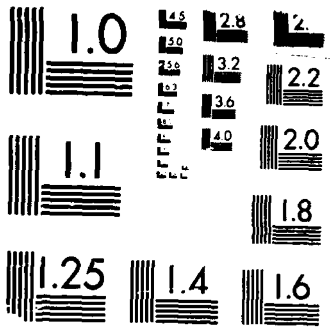

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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER: AFOSR-TR- 88-0389
2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): Convex Duality Approach to the Optimal Control or Diffusions
5. TYPE OF REPORT & PERIOD COVERED: Reprint
6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Wendell H. Fleming and Domokos Vermes
8. CONTRACT OR GRANT NUMBER(s): AFOSR-85-0315
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Lefschetz Center for Dynamical Systems, Division of Applied Mathematics, Brown University, Providence, RI 02912
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F, 2304/A1
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research, Bolling Air Force Base, Washington, DC 20332

NM

12. REPORT DATE: January 1988
13. NUMBER OF PAGES: 25
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): same as 11
15. SECURITY CLASS. (of this report): Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release: distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

DTIC ELECTE MAY 0 4 1988 S D

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18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

INCLUDED

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S/N 0102-LF-014-6601

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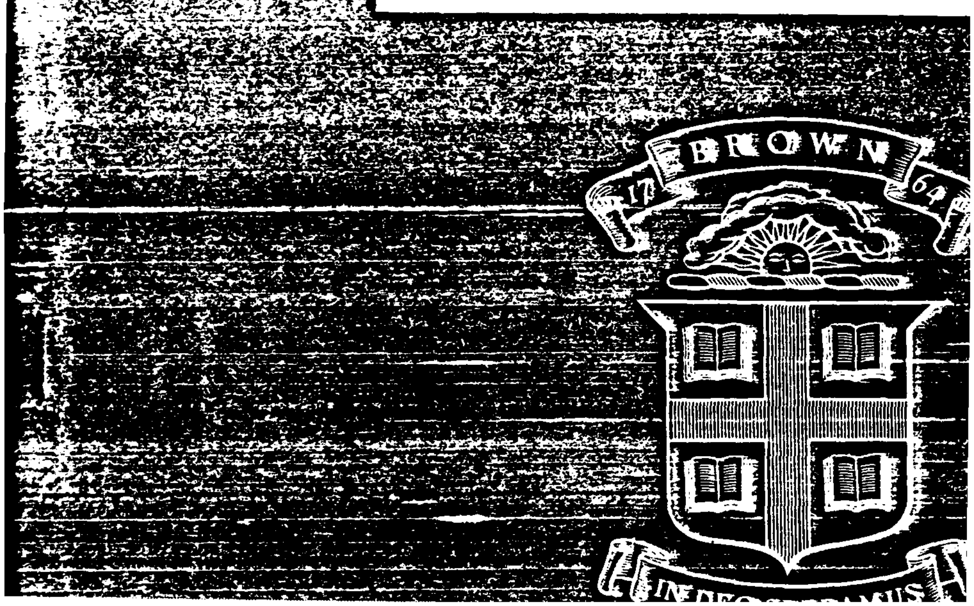

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CONVEX DUALITY APPROACH TO THE OPTIMAL CONTROL OF DIFFUSIONS

WENDELL H. FLEMING† AND DOMOKOS VERMES‡

1. INTRODUCTION

We consider $\mathbf{R}^n$-valued diffusion processes governed by the stochastic differential equation

$$dx_s = b(s, x_s, u_s)ds + \sigma(s, x_s, u_s)dw_s, \qquad x_t = x \tag{1.1}$$

with w_s an $\mathbf{R}^n$-valued Brownian motion and u_s a non-anticipative $Y \subset\subset \mathbf{R}^n$-valued control process. The objective is to minimize the expected (possibly discounted) cost

$$J^u(t,x) := \mathsf{E}^u_{t,x} \int_t^T e^{-c(s,x_s,u_s)} l(s, x_s, u_s)\, ds \tag{1.2}$$

over all control processes u. Here T is a finite or infinite planning horizon. Additional terminal costs could also be included.

An important feature of the present paper is that we do not make any ellipticity assumption, the matrix σ can be degenerate or even identically zero. This means the approach covers both deterministic and stochastic control theory.

Another specialty is that the running cost (and terminal cost if present) is not required to be bounded or continuous, merely lower semi-continuous and of polynomial growth. This makes it possible, among other things to include also problems where the objective is e.g. to minimize the probability of the event that the state ever leaves a closed subset of the state space or to maximize the hitting probability of a target set; and in particular to cover the fixed end-point problem of deterministic control theory.

In distinction from most papers in the field, the present approach does not use dynamic programming but is based on duality of convex analysis. We embed our control problem into a convex mathematical programming problem on a space of measures and consider its dual which turns out to involve the Hamilton-Jacobi-Bellman (HJB) equation. More precisely we find that the dual of the original minimization problem is to seek the supremum of all smooth subsolutions of the Hamilton-Jacobi-Bellman equation. From the existence of an equilibrium point for the primal-dual game it then follows, in particular, that the optimal value function is the upper envelope of the smooth subsolutions of the Hamilton-Jacobi-Bellman equation.

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*This research was supported in part by the Institute for Mathematics and its Applications with funds provided by the NSF and Office of Naval Research.

† Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912. Partially supported by NSF under grant MCS-8121940, by ONR under contract N00014-83-K0542 and by AFOSR under contracts F-49620-86-C-0111 and AFOSR-85-0315.

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The proof consists of two major steps. First we construct the minimization problem on the space of measures which contains the original control problem embedded (§3) and apply the Fenchel-Rockafellar duality theorem [4] to arrive at the HJB equation (§4). In the second step we prove that the embedding is actually tight: the infimum is the same both in the original and in the extended problem (§§5-6). This second part of the proof is based on the separation theorem and uses some analytic tools like mollification and Sobolev estimates, which in turn are derived by control-theoretic arguments. Roughly one could say that the separation is carried out by a sufficiently smooth control problem.

The usefulness of the duality theorem in control theory was first demonstrated by Vinter and Lewis [6], [7] who proved similar results for deterministic control problems. Their approach was made available for stochastic control problems in [5] by basing it on the theory of occupation (potential and harmonic) measures and infinitesimal operators. The present paper extends the method to the optimal control of diffusions. Since the diffusion matrix is allowed to degenerate, the presented results apply uniformly to both deterministic and stochastic control problems. The novel proof of the tightness of the embedding is not only more general but even in the classical deterministic case it is more direct than the arguments of [6].

In [3] Lions characterizes the optimal value function of stochastic control as the largest generalized subsolution of the Hamilton-Jacobi-Bellman equation. The approach and method of proof differs from the one followed here.

2. Formulation of the Problem

Let T be the planning horizon, either a non-negative number or $+\infty$. We take $0 \leq t \leq T$. If $T < \infty$ then the state space will be $E^0 := [0,T] \times \mathbf{R}^n$ and if $T = +\infty$ then $E^0 := [0,T) \times \mathbf{R}^n$. We denote by E the one-point compactification of E^0 and introduce the notation $S^0 := E^0 \times Y$ and $S := E \times Y$. Note that E and S are compact.

The coefficients $\sigma(t,x,y)$ and $b(t,x,y)$ as well as the discount rate $c(t,x,y) \geq 0$ are assumed to be bounded continuous functions on S^0 such that their first partial derivatives with respect to t and second partial derivatives with respect to x exist and, together with the functions themselves can continuously be extended to S. The running cost l is assumed to be lower semi-continuous on S and of at most polynomial growth. The case of additional terminal costs will be considered in §8.

For simplicity we assume that either the planning horizon T is finite or that there is a strict discounting. i.e. $c_0 = \inf_{\sigma \in S^0} c(\sigma) > 0$. The effect of the discounting will be included into the process as an exponential killing or a jump to the fictitious isolated cemetery state Δ at the killing time Θ. In what follows all expectation signs E will refer to the killed process. The only exception is the sans serif E in formula (1.2) which denotes the expectation of the non-killed process, i.e.

$$E\Phi(x_r) = E\Phi(x_r) \cdot 1_{\{\Theta \geq r\}} = \mathsf{E}\Phi(x_r) \int_t^r e^{-c(s,x_s,u_s)}\, ds.$$

We will also use the notation $\tau := \min(\Theta, T)$ and refer to it as the life-time of the processes. The cost J^u can then be expressed in the three equivalent forms

$$J^u(t,x) = E^u_{t,x} \int_t^T e^{-c(s,x_s,u_s)} l(s,x_s,u_s)\,ds$$
$$= E^u_{t,x} \int_t^\tau l(s,x_s,u_s)\,ds = E^u_{t,x} \int_t^T l(s,x_s,u_s)\,ds. \qquad (1.2')$$

The assumptions about the boundedness of the coefficients, growth of the costs, and boundedness of the expected life-time can be substantially relaxed. In fact, the proofs use a much less stringent but also less explicit assumption; c.f. the remark following Lemma 2.1.

The spaces of functions on S^0 and E^0 which are continuously extendable to S and E will be denoted by $C(S)$ and $C(E)$ respectively and they are considered to be Banach spaces normed by the supremum norm. In Lemma 2.1 we will introduce a continuous positive weight function $\gamma : [0,T) \times \mathbf{R}^n \to (0,\infty)$ associated with the control problem under investigation. We will consider the weighted spaces

$$C_\gamma(S) := \{f \in C(S^0) : f/\gamma \in C(S),\ \|f\|_\gamma := \sup_{\xi\in E, y\in Y} |f(\xi,y)|/\gamma(\xi) < \infty$$
$$\text{and} \lim_{|\xi|\to\infty} |f(\xi,y)|/\gamma(\xi) = 0\},$$

$C_\gamma(E)$ is defined analogously.

$$C^2_\gamma(E) := \{\Phi \in C_\gamma(E) : \Phi(T,x)/\gamma(T,x) = 0,\ \Phi_t, \Phi_{x_i}, \Phi_{x_i,x_j} \in C_\gamma(E)\ \forall\, i,j = 1,\dots,n\}.$$

In the subsequent expositions C^2_γ can always be substituted by the set of all infinitely often differentiable functions satisfying the boundary condition $\Phi(T,x)/\gamma(T,x) = 0$ and with all derivatives in $C_\gamma(E)$. We will refer to the elements of C^2_γ as smooth functions.

$\mathcal{M}^\gamma_\pm(S)$ will denote the space of all signed Borel measures M on S^0 for which the norm $\|M\|_\gamma = \int \gamma\, dM^+ + \int \gamma\, dM^-$ is finite. Here M^+ and M^- are the positive and negative parts of the Jordan decomposition of M. With obvious identification elements of $\mathcal{M}^\gamma_\pm(S)$ can be considered as signed measures on S not assigning mass to $\{\infty\} \times Y$.

If Γ is a positive constant then $\mathcal{M}^{\gamma,\Gamma}(S)$ will denote those non-negative measures from $\mathcal{M}^\gamma_\pm(S)$ for which $\|M\|_\gamma \le \Gamma < +\infty$.

The set $\mathcal{U}$ of all admissible controls consists of all Y-valued control processes u_s which are progressively measurable with respect to the filtration of the Brownian motion w_s. If $u \in \mathcal{U}$ then x^u_s denotes the solution of the stochastic differential equation (1.1) corresponding to u, satisfying the initial condition $x^u_t = x$ and killed at rate $c(\cdot)$. The corresponding expectation operator will be denoted by $E^u_{t,x}$ and if no confusion can arise the superscript u will be omitted from x^u_t inside the expectation.

With each control $u \in \mathcal{U}$ we associate the measure M^u defined on the compact space $S = E \times Y$ which is the extension of

$$
\begin{aligned}
M^u(B_t \times B_x \times B_y) &:= E^u_{t,x} \int_{[t,T]\cap B_t} 1_{B_x}(x^u_s) \cdot 1_{B_y}(u_s)\, ds \\
M(\infty \times Y) &:= 0.
\end{aligned} \tag{2.1}
$$

Here $B_t \subset [0,\infty]$, $B_x \subset \mathbf{R}^n$, $B_y \subset Y$ are arbitrary Borel sets and 1_B denotes the indicator function of the set B. Note that though the notation does not indicate it, the measures M^u depend on the initial condition $x_t = x$ in (1.1) which is considered to be fixed. We will denote the set of all such M^u corresponding to some $u \in \mathcal{U}$ by $\underline{\mathcal{M}}^S(t,x)$.

Intuitively, $M([t,t'] \times B_x \times B_y)$ measures the expected time before t' spent by the *killed* process x^u_s in the set B_x while control values from $B_y \subset Y$ were supplied. In particular, $M^u(\cdot,\cdot,Y)$ is the potential (or occupation) measure of the killed time-space process (s, x^u_s).

The infinitesimal operator of the killed Markov process x^y_t corresponding to the constant control $u_t \equiv y \in Y$ is defined for each $\Phi \in C^2(E^0)$ and is given by the expression

$$
A^y\Phi(t,x) = \frac{\partial \Phi(t,x)}{\partial t} + \sum_{i,j=1}^{n} a_{i,j}(t,x,y)\frac{\partial^2 \Phi(t,x)}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(t,x,y)\frac{\partial \Phi(t,x)}{\partial x_i} - c(t,x,y)\Phi(t,x)
$$

with $(a_{ij}) = \frac{1}{2}\sigma^T \cdot \sigma$. We will use this notation also for non-smooth functions Φ i.e. to denote the value of the expression on the right-hand side at every point (t,x) where the corresponding partial derivatives exist.

To interconnect the assumptions on discounting, termination and growth as well as to express them in a technically convenient analytic form we prove the following

LEMMA 2.1. *There exist constants $0 < \alpha < \bar{\alpha}$ and a twice continuously differentiable function $\gamma : [0,T) \times \mathbf{R}^n \to (0,\infty)$ satisfying*

$$
0 < \alpha\gamma \leq -A^y\gamma \leq \bar{\alpha}\gamma \tag{2.2}
$$

everywhere in $(0,T) \times \mathbf{R}^n$ for all $y \in Y$.

PROOF: We will construct γ separately for the discounted and for the finite horizon case.

1.) Discounted case. The infinitesimal operator of the exponentially killed process is of the form $A\Phi = D\Phi - c\Phi$ with D a (possibly degenerate) second order differential operator. We define

$$
\gamma(t,x) := (\cosh pt) \prod_{i=1}^{n} \cosh px_i \tag{2.3}
$$

with a p yet to be determined approximately.

A straightforward calculation shows that

$$
-K(p)\gamma(t,x) \leq D\gamma(t,x) \leq K(p)\gamma(t,x) \tag{2.4}
$$

with $K(p) = \sum_{i=1}^n \|a_{ij}\|p^2 + \sum_{i=1}^n \|b_i\|p = \bar{a}p^2 + \bar{b}p$. Consequently

$$K(p) + \|c\| \cdot \gamma \leq A\gamma = D\gamma - c \cdot \gamma \leq (K(p) - c) \cdot \gamma. \tag{2.5}$$

If $c_0 = \inf c > 0$ then the quadratic equation $\bar{a}p^2 + \bar{b}p - c_0 = 0$ has exactly one positive root p_0. Choosing p from the interval $(0, p_0)$ we get that $c_0 - K(p) > 0$, hence (2.2) is satisfied with $\alpha := c_0 - K(p)$ and $\bar{\alpha} := \|c\| + K(p)$.

2. Finite horizon case. We define

$$\gamma(t, x) := [1 + (T - t)] \cdot \prod_{i=1}^{n} \cosh px_i = [1 + (T - t)]\gamma_0(x).$$

Using the notation $A\Phi = \partial\Psi/\partial t + D_x\Phi$ a calculation analogous to that of the discounted case yields

$$-A\gamma(t, x) = \gamma_0(x) - (1 + T - t)D_x\gamma_0(x) \geq \left[(1 + T - 1)^{-1} - K(p)\right] \cdot (1 + T - t) \cdot \gamma_0(x). \tag{2.6}$$

With $(1 + T)^{-1}$ in place of c_0, the above argument shows that if p is chosen from $(0, p_0)$ then γ satisfies (2.2) with $\alpha := (1 + T)^{-1} - K(p) > 0$ and $\bar{\alpha} := 1 + K(p)$. The proof of the lemma is complete.

We formulate some consequences of Lemma 2.1 which will be used at various places during the subsequent expositions.

COROLLARY.

(1) $\int \gamma\, dM^u \leq \gamma(t, x)/\alpha < +\infty$ *for every* $M^u \in \underline{\mathcal{M}}^S(t, x)$. *In other words, the constant* $\Gamma := \gamma(t, x)/\alpha < +\infty$ *is a uniform upper bound for the expressions* $E^u_{t,x} \int_t^T \gamma(s, x^u_s)\, ds$ *for every process* x^u_t *generated by a control* $u \in \mathcal{U}$ *and starting from initial state* $x_t = x$.

(2) $\gamma(t, x)$ *grows asymptotically not faster than an exponential function as* $|x| \to \infty$, $t \to \infty$.

(3) For every $(t, x) \in E^0$, $\quad 0 \leq s < +\infty$ *and* $u \in \mathcal{U}$ *we have*

$$1 - e^{-\alpha s} \leq 1 - \gamma^{-1}(t, x)E^u_{t,x}\gamma(t + s, x_{t+s}) \leq 1. \tag{2.7}$$

PROOF:

(1) follows from Dynkin's formula. In fact, if $T < \infty$ we have

$$\begin{aligned}\int \gamma\, dM^u &\leq \frac{1}{\alpha}\int (-A^u\gamma)\, dM^u = \frac{1}{\alpha}E^u_{t,x}\int_t^T (-A^u\gamma)(s, x^u_s)\, ds \\ &= \frac{1}{\alpha}\left[\gamma(t, x) - E^u_{t,x}\gamma(T, x_T)\right] \leq \gamma(t, x)/\alpha.\end{aligned}$$

Since the bound is independent of T, the inequality remains true as $T \to +\infty$.

(2) is immediate from the construction of γ in the proof of Lemma 2.1.

(3) The left-hand side of (2.2) can be written as $A^u\gamma + \alpha\gamma \leq 0$. By the Feynman - Kac formula it follows that $E^u_{t,x}e^{\alpha s}\gamma(t + s, x^u_{t+s}) \leq \gamma(t, x)$ with an $\alpha > 0$. Subtracting both

sides of inequality $E^u_{t,x}\gamma(t+s, x^u_{t+s}) \leq e^{-\alpha s}\gamma(t,x)$ from $\gamma(t,x)$ gives $\gamma - E\gamma \geq \gamma(1 - e^{-\alpha s})$ which proves the left-hand side of (2.7). The right-hand side is trivial since $\gamma^{-1}E\gamma \geq 0$.

Remark. The growth, discounting and termination conditions required earlier in this section will be used in the subsequent expositions only indirectly through the statement of Lemma 2.1. Consequently all results of this paper remain valid under other sets of assumptions which assure the existence of a γ with property (2.2). Examples of other possible sets of such assumptions are

(i) Coefficients a_{ij}, b_i satisfy linear growth conditions, the discounting is strict, the running cost is bounded. In this case γ can be chosen asymptotically as $|x|^p$ with $p < C_0$ and $\alpha = C_0 - p$.

(ii) Coefficients a_{ij}, b_i satisfy linear growth conditions, the time horizon is finite, the running cost is of polynomial growth. Then one can choose $\gamma(t,x) \sim [1 + K(T-t)]|x|^p$ with an appropriate K and p.

Now we return to our original control problem. Although we assumed l to be only lower semi-continuous, in §§3-6 of the paper we will consider continuous running costs. The extension of all obtained results to the general semi-continuous case will be an additional step in §7. With the notation introduced the control problem we will consider is §§3-6 can be formulated as the

Strong Problem. For a given running cost $l \in C_\gamma(E)$ and initial state $(t,x) \in E^0$

$$\text{minimize} \quad \int l\, dM^u \quad \text{over all} \quad M^u \in \underline{\mathcal{M}}^S(t,x).$$

We can define the optimal value ψ of the strong problem as a function of the initial state

$$\psi(t,x) := \inf\Big\{ \int l\, dM^u : M^u \in \underline{\mathcal{M}}^S(t,x) \Big\}.$$

3. The weak formulation of the control problem

It follows from Ito's formula, that for arbitrary non-anticipative control process $u \in \mathcal{U}$ the generalization of the fundamental theorem of calculus (Dynkin's formula) holds true. For every twice continuously differentiable Φ we have

$$E^u_{t,x}\Phi(\sigma, x_\sigma) - \Phi(t,x) = E^u_{t,x}\int_t^\sigma A^{u_s}\Phi(s, x_s)\, ds \tag{3.1}$$

provided $\sigma \leq \tau$ is a stopping time such that the expectations exist.

If we apply this formula to the terminal time τ and to smooth functions $\Phi \in C^2_\gamma$ which vanish at the terminal state Δ then by $\Phi(\tau, x_\tau) = \Phi(\Delta) = 0$ we find that

$$-\Phi(t,x) = \int A^y \Phi(t',x') M^u(dt',dx',dy) \tag{3.2}$$

holds true for every $u \in \mathcal{U}$ whenever $A\Phi \in C_\gamma$.

We introduce the notations

$$\mathcal{M}_A(t,x) := \{M \in \mathcal{M}_\pm^\gamma(S) : -\Phi(t,x) = \int A\Phi\, dM \quad \text{for all} \quad \Phi \in C_\gamma^2(E)\}$$

and

$$\underline{\mathcal{M}}^W(t,x) := \mathcal{M}^{\gamma,\Gamma}(S) \cap \mathcal{M}_A(t,x) \quad \text{with} \quad \Gamma = \gamma(t,x)/\alpha.$$

Since for every $u \in \mathcal{M}$ the measure $M^u \in \underline{\mathcal{M}}^S(t,x)$ is in both $\mathcal{M}^{\gamma,\Gamma}(S)$ and $\mathcal{M}_A(t,x)$ our original control problem, the "Strong Problem" is embedded in the following

$$\underline{\text{Weak Problem}} \qquad \text{Minimize} \quad \int l\, dM \quad \text{over} \quad M \in \underline{\mathcal{M}}^W(t,x) \tag{3.3}$$

This is a minimization problem on the space of measures with linear objective and convex constraints. In fact if $l \in C_\gamma(S)$ then by Riesz' theorem $\int l\, dM$ is a continuous linear functional on the space of signed measures $\mathcal{M}_\pm^\gamma$. For each $\Phi \in C_\gamma^2$ relation (3.2) imposes a continuous linear restriction on M, consequently their intersection $\mathcal{M}_A(t,x)$ is a closed linear set in $\mathcal{M}_\pm^\gamma$. Finally $\mathcal{M}^{\gamma,\Gamma}$ is a w^*-compact convex subset of $\mathcal{M}_\pm^\gamma$.

The feasible set of the strong problem consists of all $M^u \in \underline{\mathcal{M}}^S$ generated by a control $u \in \mathcal{U}$ via the stochastic differential equation (1.1). This set is contained in the feasible set $\underline{\mathcal{M}}^W$ of the weak problem, thus the optimal value $\psi(t,x) := \inf\{\int l\, dM^u : u \in \mathcal{U}\}$ is not less than the minimum $\Psi(t,x) = \inf\{\int l\, dM; M \in \underline{\mathcal{M}}^W(t,x)\}$ in the weak problem. Note that the initial state (t,x) is involved in the strong problem through the initial condition (1.2) and in the weak problem through the definition of $\mathcal{M}_A(t,x)$.

In what follows, we will first characterize the value function $\Psi(t,x)$ of the weak problem by solving its dual, a maximization problem in the function space $C_\gamma(E) \subset C_\gamma(S)$. More precisely it will turn out that the dual of the minimization problem (3.3) is to find the supremum of all smooth subsolutions to the Hamilton-Jacobi equation.

To make duality methods applicable it is convenient to bring the weak problem to the Fenchel normalform. Using extended valued functions we reformulate the convexly constrained linear problem as an unconstrained convex problem. In fact, we introduce the functionals h_1 and $h_2 : \mathcal{M}_\pm^\gamma(S) \to \overline{\mathbf{R}}^1$ by

$$h_1(M) := \begin{cases} \int l\, dM & \text{if } M \in \mathcal{M}^{\gamma,\Gamma}(S) \\ +\infty & \text{otherwise} \end{cases}$$

$$h_2(M) := \begin{cases} 0 & \text{if } M \in \mathcal{M}_A(t,x) \\ -\infty & \text{otherwise.} \end{cases}$$

Both h_1 and $-h_2$ are convex and lower semi-continuous. It is immediate that the weak problem is equivalent to the following

Fenchel Problem Minimize $h_1(M) - h_2(M)$ over all $M \in \mathcal{M}^\gamma_\pm(S)$.

4. Duality and the Hamilton-Jacobi problem

Recall that the space S is compact, thus by Riesz' theorem $C^*_\gamma(S) = \mathcal{M}^\gamma_\pm(S)$. In other words $C_\gamma(S)$ and $\mathcal{M}^\gamma_\pm(S)$ are spaces in duality connected by the bilinear form.

$$\langle \phi, \mu \rangle = \int \phi \, d\mu \qquad \phi \in C_\gamma, \qquad \mu \in \mathcal{M}^\gamma_\pm \tag{4.1}$$

The norm topology of C_γ and the weak*- topology of $\mathcal{M}^\gamma_\pm$ are compatible with the pairing. the continuous linear functionals on both spaces are exactly those representable by the bilinear form. If H and h are convex real-valued functions defined on $C_\gamma(S)$ and $\mathcal{M}^\gamma_\pm(S)$ respectively then their Legendre-Fenchel transforms (convex conjugates) are defined by

$$H^*(\mu) := \sup\Big\{ \int \phi \, d\mu - H(\phi) : \phi \in C_\gamma(S) \Big\} \tag{4.2}$$

$$h^*(\phi) := \sup\Big\{ \int \phi \, d\mu - h(\mu) : \mu \in \mathcal{M}^\gamma_\pm(S) \Big\}. \tag{4.3}$$

If the original function h or H was convex and lower semi-continuous then it coincides with its double conjugate, i.e. $H^{**} = H$, $h^{**} = h$. Conjugates of concave functions are defined analogously but with inf in place of sup, and have the corresponding properties.

Now we compute the Legendre-Fenchel transforms of the functionals h_1 and h_2. We use the quantities γ and α as they were introduced in Lemma 2.1.

LEMMA (4.1). $h_1^*(\phi) = \alpha^{-1} \cdot \gamma(t,x) \cdot \|(\phi - l)^+\|_\gamma = \alpha^{-1} \cdot \gamma(t,x)$
$\cdot \sup\{[\phi(\sigma) - l(\sigma)]/\gamma(\theta,\xi)$: *over all* $\sigma = (\theta, \xi, \eta) \in S$ *such that* $\phi(\sigma) - l(\sigma) \geq 0\}$.†

PROOF:

$$h_1^*(\phi) = \sup\Big\{ \int \phi \, d\mu - h_1(\mu) : \mu \in \mathcal{M}^\gamma_\pm \Big\} = \sup\Big\{ \int (\phi - l) \, dM : M \in \mathcal{M}^{\gamma,\Gamma} \Big\}$$
$$= \sup\Big\{ \int [(\phi - l)/\gamma]\gamma \, dM : M \geq 0, \int_\gamma dM \leq \Gamma = \gamma(t,x)/\alpha \Big\}. \tag{4.4}$$

Since ϕ and l are in C_γ, the continuous function $(\phi - l)/\gamma$ attains its maximum at some point $\sigma_0 = (t_0, x_0, y_0)$ of the compact set S. If $(\phi - l)(t_0, x_0, y_0)/\gamma(t_0, x_0) > 0$ then $t_0 < \infty$, $x_0 \neq \infty$ and the sup in (3.4) can be attained by concentrating all available mass of the measure $\gamma \, dM$ to the point $\sigma_0 \in S$. We have to choose $M(ds) := \gamma(t,x)/(\alpha \cdot \gamma(t_0, x_0))\delta_{\sigma_0}(ds)$ with δ_{σ_0} denoting the Dirac measure assigning unit mass to the singleton $\{\sigma_0\}$. Then we have

†$(f)^+$ denotes the positive part of the function f, i.e. $f^+(x) = \max\{0, f(x)\}$

$$h_1^*(\phi) = \gamma(t,x)(\phi - l)(\sigma_0)/(a \cdot \gamma(t_0, x_0)) = a^{-1} \cdot \gamma(t,x)\|(\phi - l)\|_\gamma \tag{4.5}$$

provided $\sup(\phi - l) > 0$.

If $\sup(\phi - l) \leq 0$, i.e. if $\phi(\sigma) < l(\sigma)$ for all $\sigma \in S$, then the maximum of the expression (4.4) is zero and is attained for $M \equiv 0$. This together with (4.5) proves the lemma.

LEMMA (4.2).

$$h_2^*(\varphi) = \begin{cases} -\lim \Phi_i(t,x) & \textit{if } \phi = \lim_{i\to\infty} A\Phi_i \textit{ with } \Phi_i \in C_\gamma^2(E), \\ -\infty & \textit{otherwise.} \end{cases} \tag{4.6}$$

PROOF: Since h_2 is concave, $h_2^*(\phi) := \inf\{\int \phi\, dM - h_2(M) : M \in \mathcal{M}_A(t,x)\}$.

Let us first assume that $\phi = A\Phi \in C_\gamma$ with some $\Phi \in C_\gamma^2$. Then, by the definition of $\mathcal{M}_A(t,x)$ for every $M \in \mathcal{M}_A(t,x)$ we have $\int \phi\, dM = \int A\Phi\, dM = -\Phi(t,x)$. Since $\mathcal{M}_A$ is non-empty, $A\Phi_1 \equiv A\Phi_2$ implies $\Phi_1(t,x) = \Phi_2(t,x)$ and hence we have $\int \varphi\, dM = \int A\Phi\, dM = -\Phi(t,x)$ whenever $\phi = A\Phi \in C_\gamma$ with some $\Phi \in C_\gamma^2$.

Let us assume now that there exists a sequence $\Phi^k \in C_\gamma^2$ such that $\|\varphi - A\Phi^k\|_\gamma \to 0$. This means that $A\Phi^k/\gamma \to \phi/\gamma$ uniformly on S as $k \to \infty$. By this uniform convergence and the finiteness of the measure $\gamma\, dM$ for every $M \in \mathcal{M}_A(t,x) \subset \mathcal{M}_\pm^\gamma$ we have

$$\int \phi\, dM = \int \frac{\varphi}{\gamma} \cdot \gamma\, dM = \int \lim_{k\to\infty} \frac{A\Phi^k}{\gamma} \cdot \gamma\, dM = \int \lim_{k\to\infty} A\Phi^k\, dM = \lim \Phi^k(t,x)$$

independently of the particular choice of the sequence $A\Phi^k$. Since $-\lim \Phi^k(t,x)$ does not depend on $M \in \mathcal{M}_A(t,x)$, we have proved the first line of (4.6).

It remains to show that $h_2^*(\phi) = -\infty$ if ϕ is not in the $\| \ \|_\gamma$ closure of the functions $A\Phi$ with $\Phi \in C_\gamma^2$. Assume $\phi_0 \in C_\gamma$ is not in the closed subspace $W := \{\phi \in C_\gamma(S) : \lim_{k\to\infty} \|\phi - A\Phi^k\|_\gamma = 0 \text{ with } \Phi^k \in C_\gamma^2\}$. Then ϕ_0 and W can be separated by a closed hyperplane. I.e. there exists an $M' \in \mathcal{M}_\pm^\gamma(S)$ such that $\int \phi_0\, dM' < 0$ while $\int \phi\, dM' = 0$ for all $\phi \in W$. In particular we have $\int A\Phi\, dM' = 0$ for all $\Phi \in C_\gamma^2$ and consequently $M + \Theta M' \in \mathcal{M}_A(t,x)$ for every $M \in \mathcal{M}_A(t,x)$ and $\Theta \in \mathbf{R}^1$. If $\overline{M}$ denotes an arbitrary fixed element of $\mathcal{M}_A(t,x)$ then we have

$$\begin{aligned} h_2^*(\phi_0) &= \inf_{M \in \mathcal{M}_A} \int \phi_0\, dM \leq \inf_{\Theta \in \mathbf{R}^1} \int \phi_0\, d(\overline{M} + \Theta M') \\ &= \int \phi_0\, d\overline{M} + \inf_{\Theta \in \mathbf{R}^1} \Theta \cdot \int \phi_0\, dM' = -\infty. \end{aligned}$$

Here we used that by assumption $\int \phi_0\, dM' \neq 0$ and that θ can be arbitrary. This completes the proof of the lemma.

The next theorem is the main result of this section. Roughly it states that seeking the maximal solution of the Hamilton-Jacobi- Bellman equation is the dual to the weak problem formulated in the previous section. As under the current weak assumptions no smooth solutions to the Hamilton-Jacobi-Bellman equation need exist, the precise formulation of the duality relationship is the following. The value function (i.e. the minimum) of the weak problem is the upper envelope (i.e. supremum) of the smooth subsolutions of the Hamilton-Jacobi equation.

THEOREM 1.

$$\begin{aligned}\Psi(t,x) &:= \min\{\int l\,dM : M \in \mathcal{M}_A(t,x) \cap \mathcal{M}^{\gamma,\Gamma}\}\\ &= \sup\{\Phi(t,x): \quad \Phi \in C^2_\gamma, \quad A\Phi + l \geq 0\}.\end{aligned}$$

PROOF: If applied to $C^*_\gamma = \mathcal{M}^\gamma_\pm$, Rockafellar's duality theorem [4] states that

$$\min\{h_1(M) - h_2(M) : M \in \mathcal{M}^\gamma_\pm(S)\} = \sup\{h^*_2(\phi) - h^*_1(\phi) : \phi \in C_\gamma(S)\} \qquad (4.7)$$

whenever the set $\{\varphi : h^*_2(\phi) > -\infty\}$ contains a finite continuity point of h^*_1. But this condition is satisfied since h^*_1 is continuous and finite on whole C_γ and $h^*_2(\phi)$ is not identically $-\infty$, and hence (4.7) holds true.

Substituting the explicit expressions for h^*_1 and h^*_2 from Lemmas 4.1 and 4.2 into (4.7) and using the fact that $\{A\Phi : \Phi \in C^2_\gamma\}$ is dense in $\{\phi : h^*_2(\phi) > -\infty\}$ we obtain

$$\begin{aligned}\Psi(t,x) &= \min\{h_1(M) - h_2(M) : M \in \mathcal{M}^\gamma_\pm(S)\}\\ &= \sup\{\Phi(t,x) - \alpha^{-1}\cdot\gamma(t,x)\cdot\|(A\Phi + l)^-\|_\gamma : \Phi \in C^2_\gamma\}\end{aligned}$$

To conclude the proof it is sufficient to show that for every $\Phi \in C^2_\gamma$ there exists a $\Phi^\sim \in C^2_\gamma$ such that $A\Phi^\sim + l \geq 0$ and $\Phi^\sim(t,x) \geq \Phi(t,x) - \alpha^{-1}\cdot\gamma(t,x)\cdot\|(A\Phi + l)^-\|_\gamma$. Choose $\Phi^\sim := \Phi - \alpha^{-1}\cdot\gamma\|(A\Phi + l)^-\|_\gamma$. Then by Lemma 1.1 $-A\gamma \geq \alpha\gamma$ holds and consequently, we have

$$\begin{aligned}A\Phi^\sim + l &= A\Phi + l - \alpha^{-1}\cdot\|(A\Phi + l)^-\|_\gamma\cdot A\gamma \geq A\Phi + l + \gamma\cdot\|(A\Phi + l)^-\|_\gamma\\ &= A\Phi + l + \gamma\cdot\sup_{(t',x',y')\in S}|(A\Phi + l)^-(t',x',y')/\gamma(t',x')| \geq 0.\end{aligned}$$

The proof of the theorem is complete.

In a less compressed form Theorem 1 states that the weak value function Ψ is the upper envelope of all $\Phi \in C^2_\gamma(E)$ satisfying the Hamilton-Jacobi inequality.

$$\begin{aligned}\Phi_t(t,x) + \min_{y\in Y}\Big\{\sum_{i,j=1}^n a(t,x,y)\Phi_{x_ix_j}(t,x) + \sum_{i=1}^n b_i(t,x,y)\Phi_{x_i}(t,x)&\\ - c(t,x,y)\Phi(t,x) + l(t,x,y)\Big\} \geq 0.&\end{aligned} \qquad (4.8)$$

Recall that the definition of C^2_γ includes $\Phi(T,x)=0$ whenever $T<+\infty$.

The fact that $A\Phi_1 \geq A\Phi_2$ implies $\Phi_1 \leq \Phi_2$ justifies to call the functions $\Phi \in C^2_\gamma$ satisfying (4.8) subsolutions of the Hamilton-Jacobi equation.

The results of the present paragraph remain valid under much more general assumptions than those made in §2. In fact, we did not use either the finite dimensionality of the state-space or the specific properties of diffusion processes. Besides Rockafellar's duality theorem, our approach was based on the validity of Dynkin's formula, but not even the denseness of C^2_γ was exploited. Since Dynkin's formula is a special case of the "general fundamental theorem of calculus" in semigroup theory, all results of the present paragraph can be generalized to the case, when the state and control spaces are locally compact separable metric spaces and C^2_γ is substituted by a linear subset $\mathcal{L}$ of $C_\gamma(E)$. Of course, this latter change affects the definition of $\mathcal{M}_A$ and consequently the weak problem itself. But still, the dual of this new "$\mathcal{L}$-weak" problem will be the problem of finding the upper envelope of all subsolutions in $\mathcal{L}$ of the Hamilton-Jacobi-Bellman equation involving the operator A. The coincidence of the primal and dual values remain preserved too.

5. Equivalence of the strong and weak formulations

We prove the equivalence under the assumption of a special approximation property of the value function corresponding to smooth costs. In Section 6 we will show that under the assumptions of the present paper this approximability is always true.

Theorem 2. *Let $f \in C^2_\gamma(S)$ denote an arbitrary smooth "running cost" and denote F the corresponding (strong) value function.*

Suppose that every such value function F can be approximated in the $\|\ \|_\gamma$-norm by a sequence of functions $F^{(\epsilon)}$ each of which has first and second derivatives essentially bounded in γ-norm and satisfies $AF^{(\epsilon)}+f \geq 0$ a.e. as well as $F^{(\epsilon)}(T,x)=0$ whenever $T<\infty$. Then, for each $(t,x)\in E^0$ and l as in Section 2 the weak and strong formulations are equivalent; their optimal value functions coincide.

Note that Theorem 2 assumes the approximability of value functions generated by smooth costs and makes a statement about the more general control problem which involves general continuous or (later) even only lower semicontinuous running cost l.

Proof: Assume that the statement of the theorem is false, there exists an initial state (t_0,x_0) such that $\Psi(t_0,x_0) < \psi(t_0,x_0)$. This means that there exists a measure $M_0 \in \underline{\mathcal{M}}^W(t_0,x_0)\backslash \underline{\mathcal{M}}^S(t_0,x_0)$ which gives rise to a cost $\int l\,dM_0$ lower than $\psi(t_0,x_0)$ the infimum over all costs generated by controls $u \in \mathcal{U}$, i.e.

$$\int l\,dM_0 < \inf\{\int l\,dM^u : u \in \mathcal{U}\}. \tag{5.1}$$

This means that the w*-continuous linear functional $\int l\,dM$ on $\mathcal{M}^1_\pm(S)$ separates an element $M_0 \in \underline{\mathcal{M}}^W$ from the w*convex-closure of the set $\underline{\mathcal{M}}^S = \{M^u : u \in \mathcal{U}\}$. In other words $\underline{\mathcal{M}}^W$ is strictly larger than the closure of $\underline{\mathcal{M}}^S$. If this is so, then M_0 and the compact set $\underline{\mathcal{M}}^S$ can also be separated by a functional $\int f\,dM$ generated by a smooth $f \in C^2_\gamma(S)$. More precisely, since smooth functions form a dense subset in C_γ there must exist an $f \in C^2_\gamma$ such that

$$\int f\,dM_0 < \inf\{\int f\,dM^u : M^u \in \underline{\mathcal{M}}^S\}. \tag{5.2}$$

Let us introduce the strong value function F corresponding to the running cost f

$$F(t,x) := \inf\Big\{\int f\,dM : M \in \underline{\mathcal{M}}^S(t,x)\Big\} = \inf_{u\in\mathcal{U}} E_{t,x}\int_t^T f(t, x_t^u, u_t)\,dt. \tag{5.3}$$

Then, according to the assumptions of the theorem, for every $\varepsilon > 0$ there exists an $F^{(\varepsilon)}$ such that the partial derivatives $F^{(\varepsilon)}_t, F^{(\varepsilon)}_{x_i}, F^{(\varepsilon)}_{x_i x_j}$ are all defined a.e., are essentially bounded and for every $y \in Y$ the inequality

$$A^y F^{(\varepsilon)}(t,x) + f(t,x,y) \geq 0 \tag{5.4}$$

is satisfied for a.e. $(t,x) \in E$ and $\|F^{(\epsilon)} - F\|_\gamma < \epsilon$.

The generalized Dynkin's formula (3.2) cannot directly be applied to (5.4) because $F^{(\varepsilon)}$ is not smooth, it should first be approximated by C^2_γ functions. The details of this approximation are presented in the next two lemmas. Using them, the conclusion of the proof of Theorem 2 will be straightforward.

LEMMA 5.1. *For every $\delta > 0$ there exists an $F^{(\varepsilon,\delta)} \in C^2_\gamma(E)$ such that*

$$\begin{aligned} &\|F^{(\varepsilon)} - F^{(\varepsilon,\delta)}\|_\gamma < \delta, \\ &\|AF^{(\varepsilon,\delta)}\|_\gamma \leq \|AF^{(\varepsilon)}\|_\gamma + \delta, \\ \textit{and}\quad &AF^{(\varepsilon,\delta)} + f \geq -\delta\cdot\gamma \quad \textit{on} \quad [\delta, T-\delta]\times \mathbf{R}^n \times Y. \end{aligned} \tag{5.5}$$

PROOF: First we extend the definition of $F^{(\varepsilon)}$ from $[0,T]\times\mathbf{R}^n$ to $[-T,2T]\times\mathbf{R}^n$ by

$$\begin{aligned} F^{(\varepsilon)}(-s,x) &:= F^{(\varepsilon)}(0,x) \qquad s\in[0,T], x\in\mathbf{R}^n \\ F^{(\varepsilon)}(T+s,x) &:= F^{(\varepsilon)}(T-s,x) \end{aligned} \tag{5.6}$$

and the functions a, b, c, and f from $[0,T]\times\mathbf{R}^n\times Y$ to $[-T,2T]\times\mathbf{R}^n\times Y$ by reflection over 0 and T: i.e. $a(-s,x,y) = a(s,x,y)$ and $a(T+s,x,y) = a(T-s,x,y)$ if $s\in[0,T], x\in\mathbf{R}^n, y\in Y$, and similarly for b, c and f.

Note that

$$\begin{aligned} A^y F^{(\varepsilon)}(-s,x) &= -F^{(\varepsilon)}_t(s,x) + A^y F^{(\varepsilon)}(s,x) \qquad s\in[0,T], x\in\mathbf{R}^n, y\in Y \\ A^y F^{(\varepsilon)}(T+s,x) &= 2F^{(\varepsilon)}_t(s,x) - A^y F^{(\varepsilon)}(T-s,x). \end{aligned} \tag{5.7}$$

Moreover, because of the Lipschitz continuity of $F^{(\epsilon)}$ we have

$$\sup_{[-T,2T]\times\mathbf{R}^n\times Y} |AF^{(\epsilon)}/\gamma| = K$$

with some finite number K. (We reserve the notation $\|\cdot\|_\gamma$ for sup over $[0,T]\times\mathbf{R}^n\times Y$.)

Let $\rho_r(t,x)$ be a non-negative symmetric C^∞-mollifier (partition of unity) with $\int\int\rho_r(\sigma,\xi)d\sigma d\xi = 1$ and $\rho_r(\sigma,\xi) = 0$ if $|\sigma|+|\xi| > r$. If $\phi\in C([-T,2T]\times\mathbf{R}^n)$ we define $\phi*\rho_r$ on $[0,T]\times\mathbf{R}^n$ by

$$(\phi*\rho_r)(t,x) = \int\int\phi(t+\sigma,x+\xi)\rho_r(\sigma,\xi)\,d\sigma d\xi \quad \text{if} \quad 0<r<T.$$

From the second relation of (5.6) it follows that $\big(F^{(\epsilon)}*\rho_r\big)(T,x) = 0$, moreover $F^{(\epsilon)}*\rho_r$ is infinitely often differentiable on $[0,T]\times\mathbf{R}^n$ and $\|F^{(\epsilon)}*\rho_r\|_\gamma \le K$. Consequently $F^{(\epsilon)}*\rho_r \in C^2_\gamma(E)$ for every $0<r<T$.

Since by (5.4) $AF^{(\epsilon)}+f\ge 0$ holds almost everywhere on $[0,T]\times\mathbf{R}^r\times Y$, it follows that

$$(AF^{(\epsilon)})*\rho_r + f*\rho_r \ge 0 \qquad \text{on} \qquad [r,T-r]\times\mathbf{R}^n\times Y$$

We want to show that for every $\delta>0$ there exists an $r>0$ such that $F^{(\epsilon,\delta)} := F^{(\epsilon)}*\rho_r$ satisfies (5.5). Clearly, we can assume $r<\delta$, i.e. $[\delta,T-\delta]\subset[r,T-r]$ and thus it is sufficient to show that

$$\|(AF^{(\epsilon)})*\rho_r - A(F^{(\epsilon)}*\rho_r)\|_\gamma \to 0 \quad \text{and} \quad \|f*\rho_r - f\|_\gamma \to 0.$$

We have

$$\begin{aligned}
&\frac{1}{\gamma(t,x)}\big[(AF^{(\epsilon)}*\rho_r - A(F^{(\epsilon)}*\rho_r)\big](t,x,y)\\
&= \frac{1}{\gamma(t,x)}\int\int\Big\{\sum_{i,j=1}^n\big[a_{ij}(t+\sigma,x+\xi,y)-a_{ij}(t,x,y)\big]F^{(\epsilon)}_{x_i,x_j}(t+\sigma,x+\xi)\\
&\quad+\sum_{i=1}^n\big[b_i(t+\sigma,x+\xi,y)-b_i(t,x,y)\big]F^{(\epsilon)}_{x_i}(t+\sigma,x+\xi)\\
&\quad-\big[c(t+\sigma,x+\xi,y)-c(t,x,y)\big]F^{(\epsilon)}(t+\sigma,x+\xi)\Big\}\rho_r(\sigma,\xi)d\sigma d\xi\\
&\le \sum\tilde a_{ij}(r)\|F^{(\epsilon)}_{x_ix_j}\|_\gamma + \sum\tilde b_{ij}(r)\|F^{(\epsilon)}_{x_i}\|_\gamma + \tilde c(r)\|F^{(\epsilon)}\|_\gamma
\end{aligned} \tag{5.8}$$

where $\tilde a_{ij}, \tilde b_i, \tilde c$ denote the moduli of continuity of the corresponding coefficients. Since the coefficients were assumed to be uniformly continuous and the $\|\cdot\|_\gamma$–norms of $F^{(\epsilon)}_{x_ix_j}, F^{(\epsilon)}_{x_i}$ and $F^{(\epsilon)}$ are finite by the assumption of the theorem, the right-hand side of the inequality tends to zero as $r\to 0$, proving the lemma.

LEMMA 5.2.

$$\int_{[t_1,t_2]\times\mathbf{R}^n\times Y} \gamma\, dM \leq (t_2 - t_1)\cdot\gamma(t,x) \tag{5.9}$$

holds true for every $M \in \underline{\mathcal{M}}^W(t,x)$ *and* $0 \leq t_1 \leq t_2 \leq T$.

PROOF: Denote $\chi(s) := (t_2-t_1)-\int_0^s 1_{[t_1,t_2]}(\sigma)d\sigma$ and let $\chi_k : [0,T] \to \mathbf{R}^1$ be a monotonely decreasing sequence of functions which are continuously differentiable in $(0,T)$, for which $\chi_k(T) = 0$ and such that $\chi_k \searrow \chi$ and $\chi'_k \nearrow -1_{[t_1,t_2]} = \chi'$ as $k \to \infty$.

For $M \in \underline{\mathcal{M}}^W(t,x) \subset \mathcal{M}_A(t,x)$, the generalized Dynkin's formula (3.2) can be applied to the functions $\Phi_k(\sigma,\xi) := \chi_k(\sigma)\cdot\gamma(\sigma,\xi)$. Using relation

$$A\Phi_k = \gamma\cdot A\chi_k + \chi_k\cdot A\gamma = \gamma\cdot\chi'_k - \gamma\cdot c\cdot\chi_k + \chi_k\cdot A\gamma$$

and the fact that $c\cdot\gamma$, $-A\gamma$, and M are non-negative, we obtain

$$\begin{aligned}
(t_2 - t_1)\cdot\gamma(t,x) \geq \gamma(t,x)\cdot\chi(t) &= \lim_{k\to\infty}\gamma(t,x)\cdot\chi_k(t)\\
&= \lim_{k\to\infty}\Phi_k(t,x) = \lim_{k\to\infty}\int -A\Phi_k\, dM\\
&= \lim_{k\to\infty}\int -\chi'_k\cdot\gamma\, dM + \int \chi_k c\cdot\gamma\, dM + \int\chi_k(-A\gamma)\, dM\\
&\geq \lim_{k\to\infty}\int -\chi'_k\cdot\gamma\, dM = \int 1_{[t_1,t_2]}\cdot\gamma\, dM
\end{aligned}$$

proving the lemma.

CONCLUSION OF THE PROOF OF THEOREM 2:

Since $F^{(\epsilon,\delta)} \in C^2_\gamma(E)$ and $M_0 \in \underline{\mathcal{M}}^W \subset \mathcal{M}_A(t,x)$, we can apply the generalized Dynkin's formula and obtain by (5.5) and (5.9)

$$\begin{aligned}
F^{(\epsilon,\delta)}(t,x) &= -\int_{[0,T]\times\mathcal{R}^n\times Y} AF^{(\epsilon,\delta)}\, dM_0\\
&\leq \int_{[\delta,T-\delta]\times\mathbf{R}^n\times Y} f\, dM_0 + \delta\cdot\int_{[\delta,T-\delta]\times\mathbf{R}^n\times Y}\gamma\, dM_0\\
&\quad + \|AF^{(\epsilon,\delta)}\|_\gamma\cdot\int_{([0,\delta]\cup[T-\delta,T])\times\mathbf{R}^n\times Y}\gamma\, dM_0.
\end{aligned}$$

Since $M_0 \in \underline{\mathcal{M}}^W \subset \mathcal{M}^{\gamma,\Gamma}$, we have $0 \leq \int_S \gamma\, dM_0 \leq \Gamma$. From Lemma 5.2 it follows that the integral in the last term is not greater than $2\delta\cdot\gamma(t,x) = 2\delta\cdot\alpha\cdot\Gamma$ and since by Lemma 5.1 $\|AF^{(\epsilon,\delta)}\|_\gamma \leq \|AF^{(\epsilon)}\|_\gamma + \delta$, we have

$$F^{(\epsilon,\delta)}(t,x) \leq \int_S f\, dM_0 + \delta\cdot 2(1+\alpha\|AF^{(\epsilon)}\|_\gamma)\cdot\Gamma \tag{5.10}$$

Choosing first ε then δ sufficiently small, from $\|F - F^{\varepsilon,\delta}\| \le \varepsilon + \delta$ and relation (5.10) it follows

$$F(t,x) = \inf\left\{ \int f\, dM^u : M^u \in \underline{\mathcal{M}}^S \right\} \le \int f\, dM_0$$

in contradiction to the choice (5.2) of f as separating functional. This proves the equivalence of the strong and weak formulations.

Remark. Assumptions on the derivatives of $F^{(\varepsilon)}$ were only needed to obtain estimate (5.8). Note that since $F^{(\varepsilon)}$ is locally Lipschitzian, its first derivatives exist a.e. and are locally bounded. This fact alone is sufficient to prove the equivalence of the strong and weak problems provided the diffusion coefficients a_{ij} do not depend on t and x. In fact, in this case the terms $[a_{ij}(t+\sigma, x+\xi, y) - a_{ij}(t,x,y)]$ are zero and no assumptions on the second derivatives $F_{x_i x_j}$ are needed, shortcutting the approximation by $F^{(\varepsilon)}$ and the entire Section 6.

COROLLARY 1. *Suppose that l is of at most linear growth, i.e. $|l(t,x,y)| \le r_0 + r_1|x| + r_2 t$. If the processes are deterministic ($a_{ij} \equiv 0$) or the diffusion coefficients are independent of time and space, then the strong and weak problems are equivalent.*

PROOF: If l is of linear growth, then γ can be chosen to be $\bar{r}\cdot(1+|x|+t)$ with $\bar{r} > \max(r_0, r_1, r_2)$. Consequently f will be uniformly Lipschitzian and so will be F. Moreover F can be represented as the sum of a concave and of a smooth function, hence the first and second partial derivatives of F exist a.e. and the first partials are uniformly bounded. The corollary then follows from the previous remark.

The measure M_0 introduced in the proof of Theorem 2 could not be in the w* convex closure of $\underline{\mathcal{M}}^S(t_0, x_0)$. The argument there in fact proves:

COROLLARY 2. *$\underline{\mathcal{M}}^W(t,x)$ is the w* convex closure of $\underline{\mathcal{M}}^S(t,x)$.*

6. A SOBOLEV APPROXIMATION OF THE VALUE FUNCTION

To complete the proof of the equivalence of the strong and weak problems it remains to show that the value function generated by a smooth running cost can be approximated by $W^{1,2}_{\infty}$ function the way required by the assumptions of Theorem 2. This kind of approximability of the value function which does not use any non-degeneracy assumptions is also of independent interest in other branches of control theory unrelated to the strong and weak formulations. This section is devoted to the proof of the result.

THEOREM 3. *Let $f \in C^2_\gamma(S)$ and let F be the corresponding value function defined by (5.3). Then for every $\varepsilon > 0$ there exists a function $F^{(\varepsilon)} \in C_\gamma(E)$ with the following properties*

(a) $\|F - F^{(\varepsilon)}\|_\gamma \le \varepsilon$;

(b) The partial derivatives $F^{(\varepsilon)}_t, F^{(\varepsilon)}_{x_i}, F^{(\varepsilon)}_{x_i,x_j}$ exist almost everywhere for every $1 \le i,j \le n$ and satisfying $|||F^{(\varepsilon)}|||_\gamma \le K(\varepsilon)$ with some constant $K(\varepsilon)$ where

$$|||F^{(\varepsilon)}|||_\gamma := \|F^{(\varepsilon)}\|_\gamma + \|F_t^{(\varepsilon)}\|_\gamma + \sum_{i=1}^{n} \|F_{x_i}^{(\varepsilon)}\|_\gamma + \sum_{i=1}^{n} \|F_{x_i x_j}^{(\varepsilon)}\|_\gamma;$$

(c) $A^y F^{(\varepsilon)}(t,x) + f(t,x) \geq 0$ *for almost every* $(t,x) \in E$, *for every* y, *and* $F^{(\varepsilon)}(T,x) = 0$ *whenever* $T < \infty$.

We denote the weighted Sobolev space of all functions satisfying (b) by $W^{1,2}_{\gamma,\infty}$.

The idea of the proof is to extend the control set of the original problem by one additional "smoothing control" giving rise to an n-dimensional Brownian motion. The value function of the extended problem will then have the required smoothness properties and by charging a sufficiently high penalty for the "smoothing" its domain of application can be kept small and this way the smoothed value function can be forced to remain close to the original one.

To be more precise, let us introduce one more additional control η so that the extended control set will be $Y \cup \{\eta\}$. The process associated with η will be the standard n-dimensional Brownian motion discounted at the lowest possible rate $c_0 = \inf_{t,x,y} c(t,x,y)$ so that we have

$$E^{\eta}_{t,x}\Phi(x^{\eta}_{t+s}) = \frac{e^{-c_0 s}}{(2\pi s)^{\eta/2}} \int \Phi(t+s,\xi)\exp\left[-\frac{|\xi - x|^2}{2s}\right] d\xi =: (\beta_s * \Phi)(t,x).$$

The infinitesimal operator corresponding to the exponentially killed Brownian motion is

$$A^{\eta}\Phi = \frac{1}{2}\Delta\Phi - c_0\Phi$$

where Δ denotes the Laplacian. Recall that in Lemma 2.1 inequality (2.2) holds not only for the family of operators $\{A^y\}_{y\in Y}$ but with possibly different numbers α and $\bar{\alpha}$ also for the extended family $\{A^y\}_{y\in Y\cup\{\eta\}}$. In particular we have $0 < \alpha\gamma \leq -A^{\eta}\gamma$.

During the period of time when the new control η is applied we charge the running cost

$$f(s,x,\eta) := L \cdot (-A^{\eta}\gamma)(s,x) = L \cdot \left(c_0\gamma(s,x) - \frac{1}{2}\Delta\gamma(s,x)\right)$$

with some constant L to be determined later. For simplicity we only allow η to be applied during at most one non-random interval of time. In other words, the extended set $\mathcal{U}_\eta$ of admissible controls will be the set of all functions of the form

$$v(\omega,s) = \begin{cases} \eta & \text{if } t_1 \leq s < t_2 \\ u(\omega,s) & \text{otherwise} \end{cases}$$

with all possible choices of $0 \leq t_1 \leq t_2 \leq T$ and $u \in \mathcal{U}$. Note that because of the possibility of killing, the processes may die before t_1 or t_2.

The value function of the extended problem can then be written as

$$F^L(t,x) := \inf_{v\in\mathcal{U}_\eta} E^v_{t,x}\int_t^T \left[f(s,x^v_s,v_s)1_Y(v_s) + L\cdot(-A^\eta\gamma)(s,x_s)1_{\{\eta\}}(v_s)\right] ds$$
$$= \inf_{\substack{t\le t_1\le t_2\le T\\ u\in\mathcal{U}}} E^u_{t,x}\Big\{\int_t^{t_1} f(s,x^u_s,u_s)\,ds$$
$$+ E^\eta_{t_1,x^u_{t_1}}\Big[\int_{t_1}^{t_2} L\cdot(-A^\eta\gamma)(s,x^\eta_s)\,ds + F(t_2,x^\eta_{t_2})\Big]\Big\}$$
$$\le \inf_{\substack{t\le t_1\le t_2\le T\\ u\in\mathcal{U}}} E^u_{t,x}\Big\{\int_t^{t_1} f(s,x^u_s,u_s)\,ds + \big(\beta_{t_2-t_1} * F\big)(t_1,x^u_{t_1})$$
$$+ L\cdot\big(\gamma - \beta_{t_2-t_1} * \gamma\big)(t_1,x^u_{t_1})\Big\}. \tag{6.1}$$

Now we show that setting the penalty L high will keep the optimal cost F^L close to F.

PROPOSITION 6.1. *For every $\varepsilon > 0$ there exists an $0 \le L_\varepsilon < \infty$ such that $\|F - F^{L_\varepsilon}\|_\gamma \le \varepsilon$.*

PROOF: $F^L \le F$ is trivial since Φ^L is the value functional of the extended control problem which contains the original problem embedded, as $t_2 = t_1$ is permitted.

To show $F - F^{L_\varepsilon} \le \varepsilon\cdot\gamma$ observe that since F/γ is bounded and uniformly continuous there exists a t_ε such the $\|F/\gamma - \beta_h * (F/\gamma)\| \le \varepsilon/2$ for all $0 \le h < t_\varepsilon$. With this t_ε let us choose $L_\varepsilon := 3\|f\|_\gamma/(\alpha\cdot t_\varepsilon)$. Now let us consider an arbitrary $(t,x)\in E$. Since F^{L_ε} is the pointwise infimum in (6.1), we can find an $\varepsilon/2$-optimal triple $\bar u$, $0 \le \bar t_1 \le \bar t_2$ i.e. such that

$$\overline{F}(t,x) := E^{\bar u}_{t,x}\Big\{\int_t^{\tau_1} f(\bar x_s,\bar u_s) + \big(\beta_h * F\big)(\bar t_1,\bar x_{\bar t_1}) + L_\varepsilon\cdot E^\beta_{\bar t_1,\bar x_{t_1}}\int_{\bar t_1}^{\bar t_2}(-A^\eta\gamma)(w_s)\,ds\Big\}$$
$$\le F^{L_\varepsilon}(t,x) + \frac{\varepsilon}{2}\cdot\gamma(t,x)$$

We use the notation $\bar x_s = x^{\bar u}_s$ and $h = \bar t_2 - \bar t_1$. Keep in mind that although $\bar u, \bar t_1, \bar t_2$ do depend on (t,x), the numbers t_ε and L_ε were chosen before (t,x) was picked, hence estimates involving only t_ε and L_ε will hold for every $(t,x)\in E$.

With the quantities just defined we can write

$$F(t,x) - F^{L_\varepsilon}(t,x) = \big[F(t,x) - \overline{F}(t,x)\big] + \big[\overline{F}(t,x) - F^{L_\varepsilon}(t,x)\big]$$
$$\le \big[F(t,x) - \overline{F}(t,x)\big] + \gamma(t,x)\cdot\frac{\varepsilon}{2} \tag{6.2}$$

and it remains to show that $F(t,x) - \overline{F}(t,x) \le \gamma(t,x)\cdot\frac{\varepsilon}{2}$.

We increase the value if we fix $\bar u$ for the initial interval $[t,\bar t_1)$ and allow minimization only after $\bar t_1$.

$$F(t,x) - \overline{F}(t,x) \le E^{\bar{u}}_{t,x}\Big\{\int_t^{\bar{t}_1} f(s,\bar{x}_s,\bar{u}_s)\,ds + F(\bar{t}_1,\bar{x}_{\bar{t}_1})\Big\} - \overline{F}(t,x)$$
$$= E^{\bar{u}}_{t,x}\Big\{\Big(\big[F - \beta_h * F\big] - L_\varepsilon \cdot \big[\gamma - \beta_h * \gamma\big]\Big)(\bar{t}_1,\bar{x}_{\bar{t}_1})\Big\}. \qquad (6.3)$$

The expression in the first bracket under the expectation sign normalized by γ can be estimated at an arbitrary $(t^1,x^1) \in E$ as

$$\frac{1}{\gamma(t',x')}\big[F(t',x') - (\beta_h * F)(t',x')\big] = \left(\frac{F}{\gamma} - \beta * \frac{F}{\gamma}\right)(t',x') + \left(\beta * \frac{F}{\gamma} - \frac{\beta * F}{\gamma}\right)(t',x')$$
$$\le \left\|\frac{F}{\gamma} - \beta * \left(\frac{F}{\gamma}\right)\right\| + \|F\|_\gamma \cdot \left(1 - \frac{\beta_h * \gamma(t',x')}{\gamma(t',x')}\right)$$

Consequently if we divide the whole expression under the expectation in (6.3) by γ we obtain for it

$$\chi(t',x') := \frac{1}{\gamma(t',x')} \cdot \Big(\big[F - \beta_h * F\big] - L_\varepsilon \cdot \big[\gamma - \beta_h * \gamma\big]\Big)(t',x')$$
$$\le \left\|\frac{F}{\gamma} - \beta_h * \frac{F}{\gamma}\right\| - (L_\varepsilon - \|F\|_\gamma) \cdot \left(1 - \frac{\beta_h * \gamma(t',x')}{\gamma(t',x')}\right). \qquad (6.4)$$

Now there are two possibilities: either $h \le t_\varepsilon$ or $h > t_\varepsilon$. If $h \le t_\varepsilon$ then by the definition of t_ε we have $\|F/\gamma - \beta_h * (F/\gamma)\| \le \varepsilon/2$. Since $L_\varepsilon \ge \|f\|_\gamma/\alpha \ge \|F\|_\gamma$ and $\gamma \ge \beta_h * \gamma$, the last term is non-negative; we may subtract it and we get $\chi(t',x') \le \varepsilon/2$ for arbitrary $(t',x') \in E$.

On the other hand if $h > t_\varepsilon$, then $1 - e^{-\alpha h} \ge 1 - e^{\alpha t_\varepsilon}$ thus by Corollary 3 to Lemma 2.1 and the choice of L_ε we have

$$(L_\varepsilon - \|F\|_\gamma) \cdot \left(1 - \frac{\beta_h * \gamma(t',x')}{\gamma(t',x')}\right) \ge \frac{2\|f\|_\gamma}{\alpha(1 - e^{-\alpha t_\varepsilon})} \cdot (1 - e^{-\alpha h}) \ge \frac{2\|f\|_\gamma}{\alpha} \ge 2\|F\|_\gamma.$$

Since both F/γ and $\beta_h * (F/\gamma)$ are bounded by $\|F\|_\gamma$, from (6.4) we find that $\chi(t',x') \le 0 \le \varepsilon/2$ for every $(t',x') \in E$.

Substituting this result back in (6.3) we find by Corollary 3 to Lemma 2.1 that

$$F(t,x) - \overline{F}(t,x) \le E^{\bar{u}}_{t,x}\gamma(t_1,\bar{x}_{t_1}) \cdot \chi(t_1,\bar{x}_{t_1}) \le \frac{\varepsilon}{2} E^{\bar{u}}_{t,x}\gamma(t_1,\bar{x}_{t_1}) \le \frac{\varepsilon}{2} \cdot \gamma(t,x).$$

This together with (6.2) gives $F - F^{L_\varepsilon} \le \varepsilon \cdot \gamma$ which completes the proof of the proposition.

It is well-known (cf. e.g. [2], Th. 4.2) that under the conditions of Theorem 3 the value function F^L permits the decomposition $F^L = \widetilde{F}^L + \widehat{F}^L$ where $\widetilde{F}^L \in C_\gamma$ is smooth, its partial derivatives $\widetilde{F}^L_t, \widetilde{F}^L_{x_i}, \widetilde{F}^L_{x_i x_j}$ belong to C_γ while $\widehat{F}^L \in C_\gamma$ is concave in x and monotone in t. In fact, for every control $J^u \in C^2_\gamma$ and the infimum of continuously parameterized family of C^2_γ functions has the above decomposition property. For such functions the generalization of Alexandrov's theorem [1] holds true; for almost every (t,x) the derivatives $F^L_t, F^L_{x_i}, F^L_{x_i x_j}$ exist and satisfy

$$F^L(t+\sigma, x+\xi) = F^L(t,x) + F^L_t(t,x)\cdot\sigma + \sum F^L_{x_i}(t,x)\xi_i + \sum\sum F^L_{x_i x_j}(t,x)\xi_i\xi_j + o(|t| + |\xi|^2). \tag{6.5}$$

It is easy to see that F^L satisfies the Hamilton-Jacobi-Bellman inequality of the extended problem almost everywhere. In fact, the next proposition is only a slight modification of known results (cf. [2], [3]) which we prove here only because the easy proof makes our exposition self-contained.

PROPOSITION 6.2. *For every $y \in Y \cup \{\eta\}$*

$$A^y F^L(t,x) + f(t,x,y)\cdot 1_Y(y) + L\cdot(-A^y\gamma)(t,x)1_{\{\eta\}}(y) \geq 0 \tag{6.6}$$

for almost every $(t,x) \in E$.

PROOF: Suppose there exists a $y \in Y_0$ and a $(t_0, x_0) \in E$ from the non-exceptional set such that

$$A^{Y_0}F^L(t_0,x_0) + f(t_0,x_0,y_0) \leq -\delta < 0.$$

Then, by the continuity of the underlying processes, there exists an $s_0 > 0$ such that for all $s \leq s_0$

$$s^{-1}\Big[E^{y_0}_{t_0,x_0}F^L(t_0+s, x_s) - F(t_0,x_0)\Big] + f(t_0,x_0,y_0) \leq -\delta/2 < 0.$$

Let $u^\delta_\xi \in \mathcal{U}_\eta$ be a $\delta/3$-optimal control for the initial state (t_0+s_0, ξ) and define

$$u^0(\omega,t) := \begin{cases} y_0 & \text{if } t_0 \leq t < t_0 + s \\ u^\delta_\xi & \text{if } t \geq t_0 + s \text{ and } x_{t_0+s_0}(\omega) = \xi. \end{cases}$$

Then this control is again in $\mathcal{U}_\eta$ and will yield the cost

$$\begin{aligned} E^{u_0}_{t_0,x_0}\int_t^T f(s,x_s,u_0(s))\,ds &\leq f(t_0,x_0,y_0)\cdot s_0 + E^{y_0}_{t_0,x_0}\Big\{F^L(t_0+s, x_{t_0+s_0}) + \delta/3\Big\} \\ &\leq F(t_0,x_0) - \delta/6 < 0 \end{aligned}$$

in contradiction to the definition of F^L as the infimum over all $u \in \mathcal{U}_\eta$. The proof for $y_0 = \eta$ is the same.

Conclusion of the Proof of Theorem 3:

Let us choose $F^{(\varepsilon)} := F^{L_\varepsilon}$ according to Proposition 6.1. Then we have $\|F - F^{(\varepsilon)}\|_\gamma \leq \varepsilon$. The derivatives $F_t^{(\varepsilon)}, F_{x_i}^{(\varepsilon)}, F_{x_i,x_j}^{(\varepsilon)}$ exist almost everywhere by Alexandrov's theorem and Proposition 6.2 shows that the Hamilton-Jacobi inequality holds true for every $y \in Y$ and for almost every $(t, x) \in E$. The smooth component $\widetilde{F}^{(\varepsilon)}$ and its derivatives are in C_γ by Krylov's cited result ([2], Th. 4.2).

It remains to show that the derivatives of the concave component $\widehat{F}^{(\varepsilon)}$ are essentially bounded by $K(\varepsilon) \cdot \gamma$.

Consider the first derivative in an arbitrary direction of the (t, x)-space. By the concavity of $\widehat{F}^{(\varepsilon)}$ this directional derivative is monotone along each line parallel to the chosen direction. Suppose that this (one-dimensional) derivative function exceeds $K \cdot \gamma$ for every K. Then by its monotonicity follows that neither can its integral function be bounded by $K_1 \cdot \gamma$. But this contradicts $\widehat{F}^{(\varepsilon)} \in C_\gamma$ which follows from Proposition 6.1. Hence there must be a $K_2(\varepsilon)$ such that $|\widehat{F}_t^{(\varepsilon)}(t,x)| + \sum_{i=1}^n |\widehat{F}_{x_i}^{(\varepsilon)}(t,x)| \leq K_2(\varepsilon)\gamma(t,x)$ almost everywhere.

As for the second derivatives, $\widehat{F}_{x_i,x_j}^{(\varepsilon)}(t,x) \leq K_3(\varepsilon)$ follows from the concavity of $\widehat{F}^{(\varepsilon)}$. To show $|F_{x_i x_j}^{(\varepsilon)}| \leq K(\varepsilon) \cdot \gamma$ consider inequality (6.6) of Proposition 6.2 for $y = \eta$. This claims that

$$F_t^{(\varepsilon)} + \frac{1}{2}\Delta F^{(\varepsilon)} - c_0 F^{(\varepsilon)} \geq L_\varepsilon \cdot A^\eta \gamma \geq L_\varepsilon \cdot \overline{\alpha} \cdot \gamma$$

where the last inequality follows from the right-hand side of (2.2). Using the estimates already obtained for $F^{(\varepsilon)}, F_t^{(\varepsilon)}, \widetilde{F}_{x_i,x_j}^{(\varepsilon)}$ we get

$$\begin{aligned}\Delta \widehat{F}^{(\varepsilon)}(t,x) &\geq -2\Big(c_0\|F^{(\varepsilon)}\|_\gamma + \|F_t^{(\varepsilon)}\|_\gamma + \frac{1}{2}\sum_{ij=1}^n \|\widetilde{F}_{x_i x_j}\|_\gamma + L_\varepsilon \overline{\alpha}\Big) \cdot \gamma(t,x) \\ &= -K_4(\varepsilon)\gamma(t,x).\end{aligned}$$

This lower bound for the sum $\sum \widehat{F}_{x_i x_j}^{(\varepsilon)}/\gamma$ together with the upper bound for the individual summands $\widehat{F}_{x_i,x_j}^{(\varepsilon)}$ obtained from the concavity of $\widehat{F}^{(\varepsilon)}$ implies $\|F_{x_i,x_j}^{(\varepsilon)}\|_\gamma \leq K(\varepsilon)$ for every $1 \leq i, j \leq n$. This completes the proof of Theorem 3.

7. Semi-continuous costs

In the previous paragraphs, in particular in §3 and §4, we assumed the running cost to be continuous $l \in C_\gamma$. Now we are going to remove this assumption and allow l to be lower semi-continuous and of growth less than γ. More precisely we denote by LC_γ the set of all functions l satisfying

(i) l is lower semi-continuous

(ii) $\sup_{(\xi,y)\in S} |l(\xi,y)|/\gamma(\xi) < \infty$

(iii) $\limsup_{|\xi|\to\infty} l(\xi,y)/\gamma(\xi) = 0$.

Such functions can be represented as upper envelopes of continuous functions $l = \sup\{f : f \in C_\gamma, f \leq l\}$ or even as limits of non-decreasing sequences of C_γ functions.

The aim of the present paragraph is to show that all results proved for continuous l in the preceding paragraphs remain true for control problems with lower semi-continuous cost functions $l \in LC_\gamma$. The key tool in approximating lower semi-continuous costs by continuous ones will be the following min-max type argument.

PROPOSITION 7.1. *Suppose $l \in LC_\gamma$ and let $\mathcal{K}$ denote an arbitrary w^*-compact subset of $\mathcal{M}_\pm^\gamma(S)$. Then*

$$\inf_{\mu\in\mathcal{K}} \int l \, d\mu = \inf_{\mu\in\mathcal{K}} \sup_{f\leq l, f\in C_\gamma} \int f \, d\mu = \sup_{f\leq l, f\in C_\gamma} \inf_{\mu\in\mathcal{K}} \int f \, d\mu. \tag{7.1}$$

PROOF: Note first, that every $l \in LC_\gamma$ defines a convex, lower w^*-semi-continuous functional on $\mathcal{M}_\pm^\gamma$, hence all infima in (7.1) are attained for some elements of the w^*-compact set $\mathcal{K}$.

The monotone convergence theorem and the obvious inequality inf sup $\geq$ sup inf yield

$$I_0 := \inf_{\mu\in\mathcal{K}} \int l \, d\mu = \inf_{\mu\in\mathcal{K}} \sup_{f\leq l, f\in C_\gamma} \int f \, d\mu \geq \sup_{f\leq l, f\in C_\gamma} \inf_{\mu\in\mathcal{K}} \int f \, d\mu.$$

Let μ^f denote the measure, for which $\int f \, d\mu^f = \inf_{\mu\in\mathcal{K}} \int f \, d\mu$. To prove the proposition it is sufficient to show the existence of a $\mu^* \in \mathcal{K}$ for which

$$\sup_{f\leq l, f\in C_\gamma} \int f \, d\mu^f \geq \int l \, d\mu^* \tag{7.2}$$

holds true.

Let f_k denote a monotone non-decreasing sequence of continuous functions with $f_k \in C_\gamma(S)$ and $f_k \nearrow l$ as $k \to \infty$. Since $\mathcal{K}$ is sequentially compact, one can select a subsequence k_i such that $\mu_i := \mu^{f_{k_i}}$ converge weakly* to a limit $\mu^* \in \mathcal{K}$ as $i \to \infty$.

Let us consider the following array of reals

$$I(i,j) := \int f_{k_i} \, d\mu_j \qquad i,j = 1,2,\ldots$$

If $i' < i$ then $I(i',j) \leq I(i,j)$ because the sequence f_{k_i} is monotone non-decreasing. The measure μ_i is by definition minimizing $\int f_{k_i} \, d\mu$ and as $f_{k_i} \leq l$, we have

$$I(i,i) = \int f_{k_i} \, d\mu_i = \inf_{\mu\in\mathcal{K}} \int f_{k_i} \, d\mu \leq \inf_{\mu\in\mathcal{K}} \int l \, d\mu = I_0.$$

Consequently all elements $I(i,j)$ with $i \leq j$ (i.e. above the diagonal) are uniformly bounded by I_0. From the monotonicity of the sequence f_{k_i} it follows that the diagonal sequence $I(i,i)$ is monotone non-decreasing and so $I_\infty := \lim_{i\to\infty} I(i,i) \leq I_0$ exists.

Since f_{k_i} is continuous and $\mu^* = w^*\text{-}\lim \mu_j$, it follows that the sequence $I(i,j)$ converges for any fixed i to a limit $I(i,\infty) = \int f_{k_i} \, d\mu^*$ as $j \to \infty$. From $I(i,j) \leq I(j,j)$ for $i \leq j$ it follows

$$I(i,\infty) = \lim_{j\to\infty} I(i,j) \le \lim_{j\to\infty} I(j,j) = I_\infty.$$

Recall that the sequence f_k was chosen such a way that $f_k \nearrow l$. Consequently the monotone convergence theorem yields

$$I(i,\infty) = \int f_{k_i}\, d\mu^* \nearrow \int l\, d\mu^* \le I_\infty = \sup_{f\le l, f\in C_\gamma} \int f\, d\mu^f.$$

In other words μ^* satisfies (7.2) and the proof is complete.

THEOREM 4. *Suppose $l \in LC_\gamma$. Then the (strong) value function of the stochastic control problem formulated in §§1-2 is the upper envelope of the smooth subsolutions of the Hamilton-Jacobi-Bellman equation, i.e.*

$$\Psi(t,x) = \sup\{\Phi(t,x) : \Phi \in C^2_\gamma, A\Phi + l \ge 0\}. \tag{7.3}$$

PROOF: It was shown in §§5-6 that $\underline{\mathcal{M}}^W$ is the closed convex hull of $\underline{\mathcal{M}}^S$. Since $\int l\, dM$ is a convex, lower w^*-semi-continuous functional on $M^\gamma_\pm$ whenever $l \in LC_\gamma$ it follows that its infimum over $\underline{\mathcal{M}}^S$ is the same as its minimum attained in $\underline{\mathcal{M}}^W$. Consequently the strong and weak value functions coincide even if l is only lower semi-continuous $l \in LC_\gamma$.

We know from Theorem 1 that the value function permits representation (7.3) if l is continuous ($l \in C_\gamma$). Proposition 7.1 can be applied to $l \in LC_\gamma$ and $\mathcal{K} = \underline{\mathcal{M}}^W$ as $\underline{\mathcal{M}}^W$ is a w^*-compact set and we obtain

$$\begin{aligned}\Psi(t,x) &= \inf_{M\in\underline{\mathcal{M}}^W(t,x)} \int l\, dM = \sup_{f\le l, f\in C_\gamma}\ \inf_{M\in\underline{\mathcal{M}}^W} \int f\, dM \\ &= \sup_{f\le l, f\in C_\gamma} \sup\{\Phi(t,x) : \Phi\in C^2_\gamma(E), A\Phi + f \ge 0\} \\ &\le \sup\{\Phi(t,x) : \Phi \in C^2_\gamma, A\Phi + l \ge 0\}.\end{aligned}$$

The opposite inequality is immediate, since for every $\Phi \in C^2_\gamma$ with $A\Phi + l \ge 0$ and for every $M \in \underline{\mathcal{M}}^W = \mathcal{M}^\Gamma_\gamma \cap \mathcal{M}_A$ Dynkin's formula yields

$$\Phi(t,x) = \int (-A\Phi)\, dM \le \int l\, dM.$$

Taking infimum over $M \in \underline{\mathcal{M}}^W$ gives $\Phi(t,x) \le \inf_{M\in\underline{\mathcal{M}}^W} \int l\, dM = \psi(t,x)$ for every $\Phi \in C^2_\gamma$ with $A\Phi + l \ge 0$ which completes the proof of the Theorem.

8. Inclusion of terminal penalties

In this final paragraph we explain how to extend the main results of the paper to problems where the cost function includes also an additional terminal penalty, i.e. where the objective is to minimize the functional

$$J^u(t,x) = E^u_{t,x}\Big\{\int_t^T l(t,x_t,m_t)\,dt + L(x_T)\Big\} \qquad (T<\infty) \tag{8.1}$$

over all controls $u \in \mathcal{U}$. Here both l and L are lower semi-continuous functions of growth less than γ at infinity. This will extend the scope of the results to include problems like the maximization of the hitting probability of a closed target set or the fixed end-point problem of deterministic control theory which were beyond the reaches of the other approaches to the Hamilton-Jacobi theory.

The key to the extension is to consider a more elaborate state space $\widetilde{S}$ which is composed of S_0, the compactification of the "interior" of the state-space, and of S_∂, the compactified "terminal boundary," as two separate components. More precisely let $\widetilde{S}$ denote the compact metric space which consists of the two isolated subsets $S_0 := E \times Y$ and $S_\partial := \overline{\mathbf{R}}^n$. Note that S_0 is the same space which was denoted by S in §2.

Every continuous function $\Phi \in C_\gamma(\widetilde{S})$ will then correspond to the pair $\Phi|_{S_0} \in C_\gamma(S_0)$ and $\Phi|_{S_\partial} \in C_\gamma(S_\partial)$ where $C_\gamma(S_0)$ is $C_\gamma(S)$ of §2 and

$$C_\gamma(S_\partial) := \{\phi \in C(\mathbf{R}^n) : \sup_{x\in\mathbf{R}^n} |\phi(x)|/\gamma(T,x) < \infty$$

and

$$\lim_{|x|\to\infty} |\phi(x)|/\gamma(T,x) = 0\}.$$

$LC_\gamma(\widetilde{S})$ will denote the set of all lower semi-continuous functions on $\widetilde{S}$, i.e. those which can be represented as upper envelopes of families of $C_\gamma(\widetilde{S})$ functions. The dual space to $C_\gamma(\widetilde{S})$ will be the set $\mathcal{M}^\gamma_\pm(\widetilde{S})$ of all pairs of measures $M = (M_0, M_\partial)$ with $M_0 \in \mathcal{M}^\gamma_\pm(S_0), M_\partial \in \mathcal{M}^\gamma_\pm(S_\partial)$ provided with the norm $\|M\|_\gamma = \int \gamma(t',x')|M_0(dt',dx',dy)| + \int \gamma(T,x)|M_\partial(dx)|$. The set of all non-negative measures $M \in \mathcal{M}^\gamma_\pm(\widetilde{S})$ with $\|M\|_\gamma \le \Gamma < +\infty$ will be denoted by $\mathcal{M}^{\gamma,\Gamma}_\pm(\widetilde{S})$.

Observe that the function

$$\tilde{l}(\sigma) := \begin{cases} l(\tau,\xi,y) & \text{if } \sigma = (\tau,\xi,y) \in S_0 \\ L(x) & \text{if } \sigma = x \in S_\partial \end{cases}$$

is in $LC_\gamma(\widetilde{S})$. The measure $\widetilde{M}^u$ defined on the Borel sets B of $\widetilde{S}$ by

$$\widetilde{M}^u(B) := \begin{cases} M^u(B) & \text{of (2.1) if } B \subset S_0 \\ P^u_{t,x}(x^u_T \in B) & \text{if } B \subset S_\partial \end{cases}$$

is in $\mathcal{M}^{\gamma,1+T}(\widetilde{S}) \subset \mathcal{M}^\gamma_\pm(\widetilde{S})$. With this notation the (strong) optimal control problem with both running and terminal costs can be formulated as follows

$$\text{Minimize} \quad \int \tilde{l}\, d\widetilde{M}^u \quad \text{over all} \quad u \in \mathcal{U}. \tag{8.2}$$

Let $C^2_\gamma(\widetilde{E})$ denote the set of all twice continuously differentiable functions Φ defined on $[0,T) \times \mathbf{R}^n$ for which $\Phi, \Phi_t, \Phi_{x_i}, \Phi_{x_i x_j}$ are all in $C_\gamma(E)(i,j = 1,2,\dots,n)$. The difference to the definition of $C^2_\gamma(E)$ in §2 is that now we do not require functions to vanish on the exit boundary $\{T\} \times \mathbf{R}^n$ for $T < \infty$. Recall that for every $\Phi \in C^2_\gamma(\widetilde{E})$ Dykin's formula

$$E^u_{t,x}\Phi(T, x_T) - \Phi(t,x) = E^u_{t,x}\int_0^T A^{u_s}\Phi(s, x_s)\, ds \tag{8.3}$$

holds true. If we introduce the operator $\tilde{A} : C^2_\gamma(\widetilde{E}) \to C_\gamma(\widetilde{S})$ by

$$\tilde{A}\Phi(\sigma) = \begin{cases} A^y\Phi(t,x) & \text{if } \sigma = (t,x,y) \in S_0 \\ -\Phi(T,x) & \text{if } \sigma = x \in S_\sigma \end{cases}$$

then with the above notation Dynkin's formula can be written in the more compact form

$$-\Phi(t,x) = \int_{\widetilde{S}} \tilde{A}\Phi\, d\widetilde{M}^u. \tag{8.4}$$

The weak problem corresponding to (8.2) can be formulated as

$$\text{Minimize} \quad \int \tilde{l}\, d\widetilde{M} \quad \text{over all} \quad \widetilde{M} \in \mathcal{M}^{\gamma,1+T}(\widetilde{S}) \cap \mathcal{M}_{\tilde{A}}(t,x)$$

with $\mathcal{M}_{\tilde{A}}(t,x) := \{\widetilde{M} \in \mathcal{M}^\gamma_+(\widetilde{S}) : \text{for which (8.4) holds } \forall \Phi \in C^2_\gamma(\widetilde{E})\}$.

All the expositions of §§3-7 can be repeated word by word for this extended notation and we obtain

THEOREM 5. *The dual to the problem (8.2) is to find the supremum of all smooth subsolutions of the Hamilton-Jacobi-Bellman equation, and we have*

$$\begin{aligned}
\Psi(t,x) &= \inf_{u \in \mathcal{U}} E^u_{t,x}\Big\{\int_t^T l(s, x_s, u_s)\, ds + L(x_T)\Big\} \\
&= \sup\Big\{\Phi(t,x) \quad \textit{over all} \quad \Phi \in C^2_\gamma(\widetilde{E}) \quad \textit{satisfying} \\
&\qquad \inf_{y \in Y} A^y\Phi(\tau,\xi) + l(\tau,\xi,y) \geq 0 \quad \textit{if} \quad 0 < \tau < T, \xi \in \mathbf{R}^n \\
&\qquad \textit{and} \quad \Phi(T,\xi) \leq L(\xi) \quad \xi \in \mathbf{R}^n\Big\}.
\end{aligned}$$

References

1. A. D. Aleksandrov, *Almost everywhere existence of second derivatives of a convex function and related properties of convex surfaces*, Sci. Notes Leningrad State Univ. **37(6)** (1939), 3–35.
2. N. V. Krylov, *Some new results from the theory of controlled diffusion processes*, Mat. Sbornik **109** (1979), 146–164.
3. P.L. Lions, *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equation I: The dynamic programming principle and applications*, Comm. in Partial Differential Eqns. **8** (1983), 1101–1174.
4. R. T. Rockafellar, *Extension of Fenchel's duality theorem for convex functions*, Duke Math. J. **33** (1966), 81–89.
5. D. Vermes, *Optimal control of piecewise deterministic Markov processes*, Stochastics **14** (1985), 165-208.
6. R. B. Vinter and R. M. Lewis, *The equivalence of strong and weak formulations for certain problems in optimal control*, SIAM J. Control **16** (1978), 546–570.
7. R. B. Vinter and R. M. Lewis, *A necessary and sufficient condition for optimality of dynamic programming type, making no a priori assumptions on the controls*, SIAM J. Control **16** (1978), 571–583.

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